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Discussion

A view to the new perturbation technique valid for large parameters

Nestor E. Sanchez

Department of Mechanical Engineering and Biomechanics, The University of Texas at San Antonio, San Antonio, TX 78249, USA

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1. Introduction

A number of publications have been generated on the basis that new techniques are now available to accurately solve nonlinear differential equations using perturbation techniques [1–3]. The basic problem can be portrayed as a linear ordinary differential equation (ODE) plus a nonlinear term that is multiplied by a constant and added to the ODE. The assumption that is usually made is that the constant term is only a perturbation of order $O(\varepsilon)$ to the linear equation. However, publications like the ones by He [1,2] claim that the solution that he has presented is an accurate solution even for very large constant ε. In these papers, He [1,2] obtains results on the lengths of the periods based on $\varepsilon \to \infty$. This proposed technique is implemented and evaluated for the same values done on the original paper [1]. It will be shown that the error of the function is huge for large values of ε. The same equation is treated with a basic Method of Multiple Scales [4,5], and then the results show the performance of the two perturbation solutions are very similar, at the point where they are valid. The reference will be an accurate numerical solution that can be found with minimum error.

E-mail address: nsanchez@utsa.edu (N.E. Sanchez).

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2. Basic equation

The original problem is

$$\ddot{u} + u + \varepsilon u^3 = 0, \quad u(0) = A, \quad \dot{u}(0) = 0. \tag{1}$$

This basic problem is changed by He [1] into another case by using

$$\beta^2 + \varepsilon\eta = 1, \tag{2}$$

to substitute Eq. (2) into Eq. (1) to obtain the equation

$$\ddot{u} + \beta^2 u + \varepsilon(u^3 + \eta u) = 0, \quad u(0) = A, \quad \dot{u}(0) = 0. \tag{3}$$

3. Higher order approximations

He [1] assumes that β^2 and u can be written as

$$\beta^2 = \omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots, \quad u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots. \tag{4,5}$$

Substituting these parameters into Eq. (3) and keeping terms to only order $O(\varepsilon^2)$, this equation can be separated into the following equations according to their order, implying $\varepsilon \ll 1$:

$$\ddot{u}_0 + \omega_0^2 u_0 = 0, \tag{6}$$

$$\ddot{u}_1 + \omega_0^2 u_1 = -\omega_1 u_0 - u_0^3 - \eta u_0, \tag{7}$$

$$\ddot{u}_2 + \omega_0^2 u_2 = -\omega_1 u_1 - \omega_2 u_0 - 3u_0^2 u_1 - \eta u_1. \tag{8}$$

The first equation can have a nice linear solution,

$$u_0(t) = A \cos \omega_0 t_0. \tag{9}$$

This can be substituted into the right-hand side (RHS) of Eq. (7) to provide

$$\ddot{u}_1 + \omega_0^2 u_1 = -(\tfrac{3}{4}A^3 + \omega_1 A + \eta A)\cos \omega_0 t - \tfrac{1}{4}A^3 \cos 3\omega_0 t. \tag{10}$$

The first term on the RHS is going to provide what is called a secular term that needs to be set to zero. The second term on the RHS is going to provide the real excitation to this equation. Therefore, this implies: $-(\tfrac{3}{4}A^3 + \omega_1 A + \eta A) = 0$. Then

$$\omega_1 = -(\eta + \tfrac{3}{4}A^2), \tag{11}$$

$$u_1 = \frac{A^3}{32} \cos 3\omega_0 t. \tag{12}$$

The last equation (8) then becomes

$$\ddot{u}_2 + \omega_0^2 u_2 = -\left(\omega_2 A + \frac{3A^5}{128\omega_0^2}\right) \cos \omega_0 t - \frac{3A^5}{128\omega_0^2} \cos 3\omega_0 t - \frac{3A^5}{128\omega_0^2} \cos 5\omega_0 t, \tag{13}$$

having additional initial conditions (given by He [1]):

$$u_2(0) = -\frac{A^3}{32\omega_0^2}, \quad \dot{u}_2(0) = 0. \tag{14}$$

To avoid the secular term in Eq. (13), it is required that

$$\omega_2 = -\frac{3A^4}{128\omega_0^2}. \tag{15}$$

The general solution for Eq. (13) is

$$u_2(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{A^5}{1024\omega_0^4}(3 \cos 3\omega_0 t + \cos 5\omega_0 t). \tag{16}$$

Using the initial conditions

$$u_2(0) = C_1 + \frac{A^5}{1024\omega_0^4}(3+1) = -\frac{A^3}{32\omega_0^2} \quad \text{and} \quad \dot{u}_2(0) = C_2 = 0, \tag{17}$$

renders

$$C_1 = -\frac{1}{256\omega_0^4}(A^5 + 8\omega_0^2 A^3), \tag{18}$$

Then, Eq. (16) becomes

$$u_2(t) = -\frac{1}{256\omega_0^4}(A^5 + 8\omega_0^2 A^3)\cos \omega_0 t + \frac{A^5}{1024\omega_0^4}(3 \cos 3\omega_0 t + \cos 5\omega_0 t). \tag{19}$$

Eq. (4) becomes, using Eqs. (11) and (15),

$$\beta^2 = \omega_0^2 + \varepsilon\left(-\frac{3A^2}{4} - \eta\right) - \varepsilon^2 \frac{3A^4}{128\omega_0^2}. \tag{20}$$

,

Now, using Eq. (2) it is possible to develop the following equation from Eq. (20):

$$\omega_0^2 = 1 + \frac{3}{4}\varepsilon A^2 + \frac{3\varepsilon^2 A^4}{128\omega_0^2}. \tag{21}$$

From this the natural frequency can be found:

$$\omega_0 = \sqrt{\frac{1}{2}\left(1 + \frac{3}{4}\varepsilon A^2\right) + \frac{1}{2}\sqrt{1 + \frac{3}{2}\varepsilon A^2 + \frac{21\varepsilon^2 A^4}{32}}}. \tag{22}$$

And the period is

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{\frac{1}{2}(1 + \frac{3}{4}\varepsilon A^2) + \frac{1}{2}\sqrt{1 + \frac{3}{2}\varepsilon A^2 + 21\varepsilon^2 A^4/32}}}. \tag{23}$$

Using Eq. (5), the perturbation solution can be expressed as

$$u(t) = A\cos\omega_0 t + \varepsilon\frac{A^3}{32\omega_0^2}\cos 3\omega_0 t + \varepsilon^2\left(-\frac{1}{256\omega_0^4}(A^5 + 8\omega_0^2 A^3)\cos\omega_0 t + \frac{A^5}{1024\omega_0^4}(3\cos 3\omega_0 t + \cos 5\omega_0 t)\right). \quad (24)$$

Expression (23) is identical to the one found by He [1]. The difference is only in the interpretation and the understanding of the solution. The paper by He [1] used only the period T (23) of the solution $u(t)$ to evaluate and measure if the solution is accurate (see also Ref. [3]). Function (24) itself is not evaluated so the performance is linked to just the time instead of the total function. In this paper the solution itself will be compared with the numerical solution that is obtained for three levels of $\varepsilon = [1, 10, 100]$. Later a typical perturbation solution is going to be derived, to show that they agree when ε is at a valid limit. If values over $\varepsilon \ll 1$ are used, then the basic principles for the derivation of the solution itself are violated. Eqs. (6)–(8) will not be valid, so the whole procedure does not have meaning.

4. Numerical solutions

The numerical solution is evaluated using the Runge–Kutta procedure specified in the Maple 8 language. The error is picked to be about 10^{-8}. The equation is

$$\ddot{u} + u + \varepsilon u^3 = 0, \quad (25)$$

with initial conditions (selected as $A = 1$) of

$$u(0) = 1, \quad \dot{u}(0) = 0. \quad (26)$$

The three solutions were found ($\varepsilon = 1, 10, 100$) and plotted with the perturbation solution (24) to compare the values. Fig. 1a shows the first approximation for $\varepsilon = 1$, as a solid curve (analytic-He), and the numerical result obtained from the integration of Eq. (25) is the dashed curve.

The period of the analytical function can be evaluated using Eq. (23) to be $T = 4.7317061$. In the same way, from the numerical solution this value can be traced to be $T_{\mathrm{num}} = 4.76802$. The percentage error on the value of the period is 0.76%. The error on the magnitude at $u(T)$ will be about $0.19(10^{-5})$%. This means that even for this large $\varepsilon = 1$, $u(t)$ is accurate.

Fig. 1b shows the two solutions when $\varepsilon = 10$. The period of the analytical function can be found using Eq. (23) to be $T = 2.12200420$ and using Eq. (24) to be $u(T) = 0.67920670$. In the same way from the numerical solution the maximum point can be traced to be $u_{\mathrm{num}} = 0.99999999371201$ at time $T_{\mathrm{num}} = 2.1918292$. This implies errors in time of 3.186% and in the magnitude of 32.079%. Therefore, the model shown in Fig. 1b is no longer valid! This extra large $\varepsilon = 10$ is violating the assumptions made to derive the model.

Fig. 1c shows the two solutions when $\varepsilon = 100$ The period of the analytical function can be found again from Eq. (23) to be $T = 0.70705690$ and using Eq. (24) to be $u(T) = -2.91772739$. In the same way, from the numerical solution the maximum point can be traced to be $u_{\mathrm{num}} =$

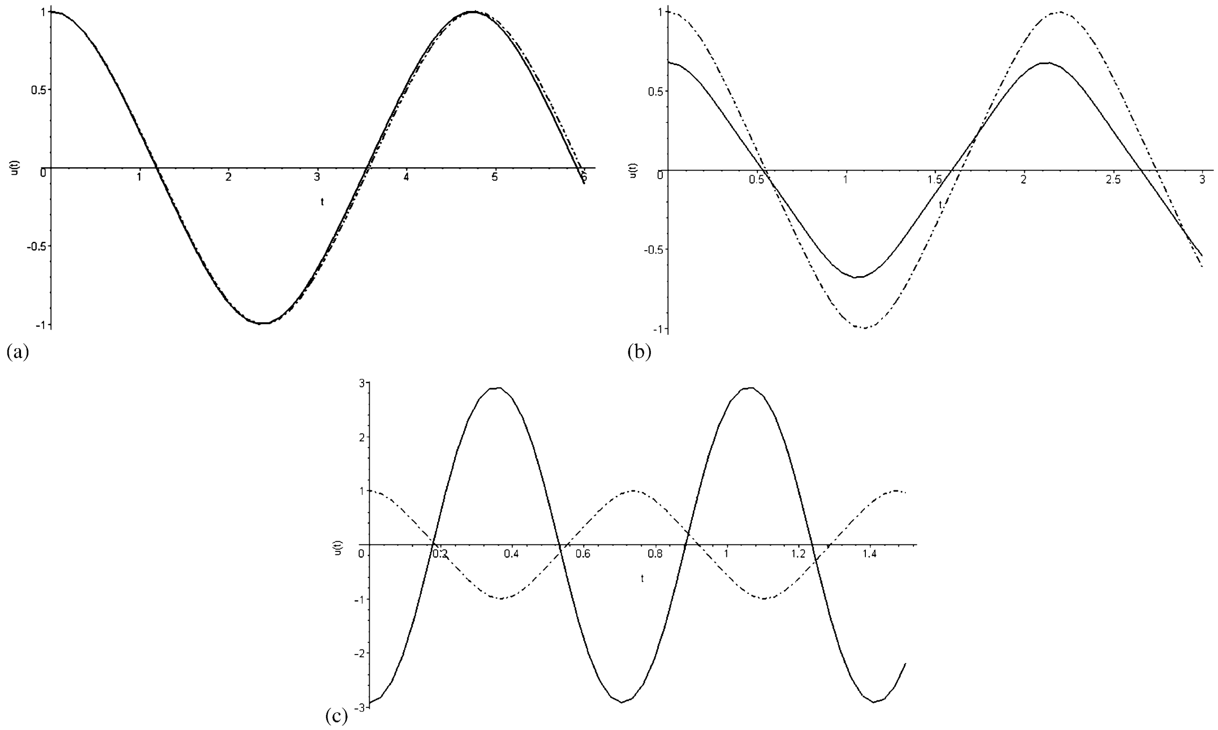

Fig. 1. Analytical solution from He (solid line) and numerical solution (dashed): (a) $\varepsilon = 1$. (b) $\varepsilon = 10$, (c) $\varepsilon = 100$.

0.9999999977820 at time $T_{\text{num}} = 0.7362890$. These imply errors in time of 3.970%, and for the magnitude, the difference is huge: 391.77%.

From the previous facts, it is easy to see that as ε grows the validity of the perturbation solution becomes poor. In particular, as ε is closed to one, the solution breaks. It also shows that even when the value for the period T is less than 5% error, as shown by He [1], the value for the function itself $u(T)$ is at a 392% error for the case $\varepsilon = 100$.

5. Perturbation using the method of multiple scales (MMS)

The original equation (1) is considered again. The following assumptions are made, $t_0 = t, t_1 = \varepsilon t, t_2 = \varepsilon^2 t$, and

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} + \cdots, \tag{27}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{\partial^2}{\partial t_0^2} + 2\varepsilon \frac{\partial^2}{\partial t_0 \partial t_1} + \varepsilon^2 \left(2 \frac{\partial^2}{\partial t_0 \partial t_2} + \frac{\partial^2}{\partial t_1^2} \right) + \cdots, \tag{28}$$

where the assumption is made that the parameter $\varepsilon \ll 1$. As presented by Nayfeh [4], the following MMS is used. Making the above substitutions into Eq. (1) and neglecting terms, of order higher than $O(\varepsilon^2)$, the following equations are found:

$$\frac{\partial^2 u_0(t_0, t_1, t_2)}{\partial t_0^2} + u_0(t_0, t_1, t_2) = 0, \tag{29}$$

$$\frac{\partial^2 u_1(t_0, t_1, t_2)}{\partial t_0^2} + u_1(t_0, t_1, t_2) = -2\,\frac{\partial^2 u_0(t_0, t_1, t_2)}{\partial t_1 \partial t_0} - u_0(t_0, t_1, t_2)^3, \tag{30}$$

$$\begin{aligned}\frac{\partial^2 u_2(t_0, t_1, t_2)}{\partial t_0^2} + u_2(t_0, t_1, t_2) = & -2\frac{\partial^2 u_0(t_0, t_1, t_2)}{\partial t_2 \partial t_0} - 2\,\frac{\partial^2 u_1(t_0, t_1, t_2)}{\partial t_1 \partial t_0} - 2\,\frac{\partial^2 u_0(t_0, t_1, t_2)}{\partial t_1^2} \\ & - 3u_0(t_0, t_1, t_2)^2 u_1(t_0, t_1, t_2).\end{aligned} \tag{31}$$

The solution of the set is the uniform approximation

$$u(t) = u_0(t_0, t_1, t_2) + \varepsilon u_1(t_0, t_1, t_2) + \varepsilon^2 u_2(t_0, t_1, t_2) + \cdots. \tag{32}$$

The solution to Eq. (29) was found to be

$$u_0(t_0, t_1, t_2) = A(t_1, t_2)\cos(t_0 + \beta(t_1, t_2)). \tag{33}$$

Substituting Eq. (33) into Eq. (30) gives

$$\begin{aligned}&\frac{\partial^2 u_1(t_0, t_1, t_2)}{\partial t_0^2} + u_1(t_0, t_1, t_2) \\ &\quad = 2\,\frac{\partial A(t_1, t_2)}{\partial t_1}\,\sin(t_0 + \beta(t_1, t_2)) \\ &\quad -\frac{3}{4}\,A(t_1, t_2)^3\,\cos(t_0 + \beta(t_1, t_2)) + 2A(t_1, t_2)\,\cos(t_0 + \beta(t_1, t_2))\frac{\partial \beta(t_1, t_2)}{\partial t_1} \\ &\quad -\frac{1}{4}\,A(t_1, t_2)^3\,\cos(3t_0 + 3\beta(t_1, t_2)).\end{aligned} \tag{34}$$

The following terms need to be eliminated for causing secular terms:

$$-\frac{3}{4}\,A(t_1, t_2)^3 + 2A(t_1, t_2)\frac{\partial \beta(t_1, t_2)}{\partial t_1} = 0, \tag{35}$$

$$2\,\frac{\partial A(t_1, t_2)}{\partial t_1} = 0. \tag{36}$$

From Eq. (36), it follows that $A(t_1, t_2) = A(t_2)$; substituting this into Eq. (35) and integrating

$$\beta(t_1, t_2) = \tfrac{3}{8}A(t_2)^2 t_1 + \beta(t_2), \tag{37}$$

Eq. (34) is left as

$$\frac{\partial^2 u_1(t_0, t_1, t_2)}{\partial t_0^2} + u_1(t_0, t_1, t_2) = -\frac{1}{4}A(t_2)^3\,\cos(3t_0 + \frac{9}{8}A(t_2)^2 t_1 + 3\beta(t_2)). \tag{38}$$

The solution of Eq. (38) is

$$u_1(t_0, t_1, t_2) = \tfrac{1}{32}A(t_2)^3\cos(3t_0 + \tfrac{9}{8}A(t_2)^2 t_1 + 3\beta(t_2)). \tag{39}$$

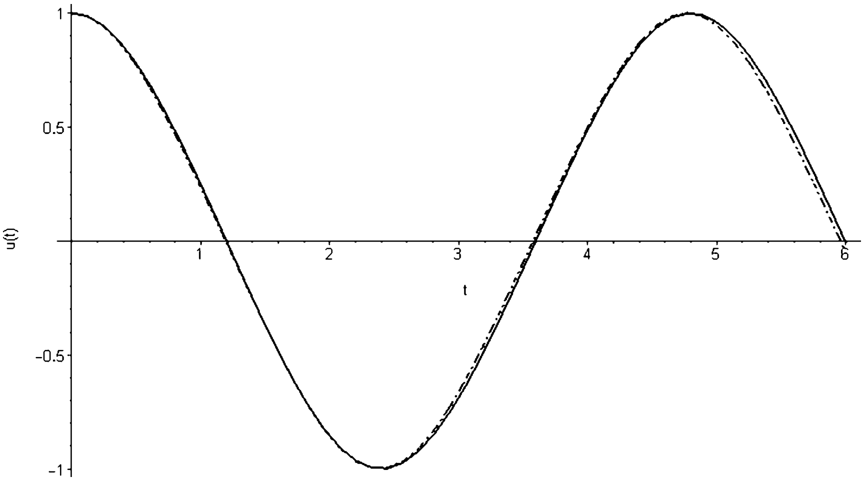

Fig. 2. Analytical solution from MMS curve (solid line) and numerical solution (dashed): $\varepsilon = 1$.

Substituting Eqs. (33) and (39) into Eq. (31) and expanding gives

$$\frac{\partial^2 u_2(t_0,t_1,t_2)}{\partial t_0^2} + u_2(t_0,t_1,t_2) = \frac{21}{128}A^5\cos\left(3t_0 + \frac{9}{8}A^2t_1 - \frac{45}{256}A^4t_2 + 3\beta_0\right) - \frac{3}{128}A^5\cos\left(5t_0 + \frac{15}{8}A^2t_1 - \frac{75}{256}A^4t_2 + 5\beta_0\right). \quad (40)$$

This is after setting the secular terms equal to zero and integrating them into

$$A(t_2) = A \quad \text{and} \quad \beta(t_2) = -\frac{15}{256}A^4t_2 + \beta_0. \quad (41)$$

At this point it is possible to set solution (32), finding the solution for Eq. (40), and going back to the original variables:

$$\begin{aligned} u(t) = {} & A\cos\left(t + \varepsilon\frac{3}{8}A^2t - \varepsilon^2\frac{15}{256}A^4t - \beta_0\right) \\ & + \varepsilon\left(\frac{1}{32}A^3\cos\left(3t + \varepsilon\frac{9}{8}A^2t - \varepsilon^2\frac{45}{256}A^4t + 3\beta_0\right)\right) \\ & - \varepsilon^2\left(\frac{21}{1024}A^5\cos\left(3t + \varepsilon\frac{9}{8}A^2t - \varepsilon^2\frac{45}{256}A^4t + 3\beta_0\right)\right) \\ & + \varepsilon^2\left(\frac{1}{1024}A^5\cos\left(5t + \varepsilon\frac{15}{8}A^2t - \varepsilon^2\frac{75}{256}A^4t + 5\beta_0\right)\right). \end{aligned} \quad (42)$$

This is the perturbation solution that was found. In order to apply it to the problem, only the initial conditions set on Eq. (1) needs to be satisfied by Eq. (42) and its derivative, and the level of ε needs to be defined. In order to get a numerical solution, again the two equations need to be solved. This set of equations is solved by Maple 8. The value of ε is taken to be beyond the point of validity at $\varepsilon = 1$. Maples gives the following answers, $A = 0.9882491$ and $\beta_0 = 0$. The difference in the amplitude turns out to be $0.2(10)^{-5}$. The period turns out to be 0.57% from the numerical integration. In essence, the same values were obtained with He's perturbation technique and with the MMS for $\varepsilon = 1$, as is shown in Figs. 2 and 1a.

6. Conclusions

A perturbation technique valid for large parameters, ε or $\varepsilon > 1$, was presented by He [1,2]. It is not an appropriate procedure and it leads to a wrong conclusion. The numerical integration shows differences in the errors in magnitude at the different ε tested ($\varepsilon = 1, 10$, and 100). It was shown that for $\varepsilon > 1$, the values predicted by the functions were not appropriate since the assumptions are that the first solution is linear and the nonlinear equation has at a higher level of the perturbation term ε. The answers obtained using the MMS perturbation procedure were the same as the ones provided by He [1] for low values of $\varepsilon \ll 1$. He presented similar developments in other publications [2], using mostly the period to show the performance of the function and not using the function itself to see if it is accurate or not. The derivation of the procedure is also very important because the terms have been ordered according to the assumed parameter ε. A large number of terms were neglected because of the assumption that $\varepsilon \ll 1$. If this is not the case, the whole procedure needs to be rewritten from the start with the proper assumptions.

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